# Circle Constructions of Polynomial Complex Roots

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**Abstract**

*Herein we consider a novel theme of analytically and geometrically constructing lines and circles through which complex roots of real quadratic, cubic, and quartic polynomial functions and special functions of higher degree can be located. When formulas are used, they are constructed using only values that are easily observed by secondary students from the graph. To engage the reader in this investigation and illustrate how students can experientially learn the material, this article contains dynamic graphing applets.*

# 1 Introduction

A common construction to locate the complex roots of a quadratic function involves the construction of a circle (shown later in Figure 2). Using the construction of circles as a motivation, the authors investigated if this could be a common theme in locating the complex roots of other polynomials.

Herein we consider real monic polynomials in the form of quadratics, cubics, quartics, and special polynomials of higher degree. Real monic polynomials have real (non-complex) coefficients in which the leading coefficient is 1. (Non-monic real polynomials can be divided by the leading coefficient to make them monic, without affecting the roots.) While these polynomials are graphed on the real Cartesian plane ($\mathbb{R} \times \mathbb{R}$), this plane does not include the non-real complex points herein under investigation. Therefore, to locate non-real complex roots, we co-label the $y$-axis with both real and imaginary values. In other words, we are identifying the complex numbers and the real plane: $\mathbb{C} = \mathbb{R} \times \mathbb{R}$. This means that $a + bi = (a, b)$. This plane retains the real-valued $x$- and $y$-axes and can locate non-real complex points in reference to the imaginary $y$-axis.

In this investigation we define *analytic constructions* as any combination of geometric constructions and analytic calculations restricted to using observable values on the graph. History is replete with examples of analytic construction regarding quadratics, cubics, and quartics (for example, [1], [6], [10], and [11]). Herein, we employ analytic constructions leading to the development of lines and circles defining the location of the complex roots for these polynomials. In most cases, these analytic constructions are rather straightforward and accomplishable by most upper secondary school students and beyond.

This paper's bibliography demonstrates that investigations of complex roots have been studied for centuries. Ferrari, Bombelli, Vieta, Simpson, Euler, Descartes, Lagrange, and Cayley all developed intricate algebraic methods for solving polynomials. (See, e.g., [5], [8], and [9].) However, all did so without the technological tools now at our disposal. While in many cases the use of technology can lead to the answering of sophisticated mathematical questions, notably, its use can also often lead to the asking of many more questions. (See, for example [2].) Inquiry of mathematical ideas through technology can lead to novel investigatory methods and possible findings. In this paper, the authors employed Maple 2016 [13], The Geometer's Sketchpad [12], and Web Sketchpad [14] to make inquiries to investigate and solve the problems at hand. The proper use of technology as a problem solving tool cannot be overvalued. In fact, the use of technology in this investigation has led to: (a) a new theorem being developed regarding quartics [3]; (b) extending the literature with unique circle and line constructions on these polynomials to locate complex roots; (c) the ability to develop a circle and line constriction on special polynomials of higher degree; (d) the development of dynamic HTML graphing apps which include Boolean conditions to consider both real and complex roots; (e) and descriptions of how to visualize the location of complex roots for quadratics, cubics, quartics, and special higher degree polynomials using minimal mathematics [4].

Readers of this paper will readily recognize its unusual form including dynamic graphs to illustrate how students can interact, experience, and learn in a tactile manner. Throughout this paper, in figures denoted as a "Dynamic Figure," the reader is invited to interact with dynamic graphs. To do so, drag the real and complex roots around the graph, change the size of the coordinate system, change a scaling factor of the function, and use hide/show buttons to display construction features. While many readers may have previously interacted with dynamic graphs through the input of variables and coefficients in polynomials in various forms, we provide a possibly novel feel by allowing readers to manipulate both real and complex roots about the screen by merely dragging these points. Furthermore, in many cases, the graphs are programmed with Boolean functions that will allow and disallow the existence of various elements; readers may wish to investigate these aspects as well. It is anticipated that providing students with this interactive experience will lead to greater understanding of the concepts in this paper.

The dynamic graphing applets in this paper allows readers to interact with sophisticated mathematical ideas without being overwhelmed by the mathematics provided in the descriptive prose. This allows readers at multiple levels of mathematical understanding to glean some ideas from this investigation commensurate with their mathematical backgrounds. Additionally, combining the dynamic graphing applets with precise, descriptive mathematics invites the reader to use this investigation as a springboard to further investigations.

One recognized technique for locating the complex roots of a quadratic polynomial is found in [7]. As seen in Figure 1, construct a tangent through the vertex of the original quadratic function $f(x)$, and reflect the parabola across this tangent. Construct a circle with diameter points at the intersection of the reflected function and the $x$-axis. Rotate the intercepts about the center of the circle by 90° to locate $(a, b\imath)$ and $(a, -b\imath)$, the complex roots. Notably, this technique employs a circle to locate the complex roots of a quadratic polynomial. Through the remainder of this investigation, we use the theme of circles and lines to locate complex roots on quadratic, cubic, and quartic polynomials and additional special polynomials of higher degree.

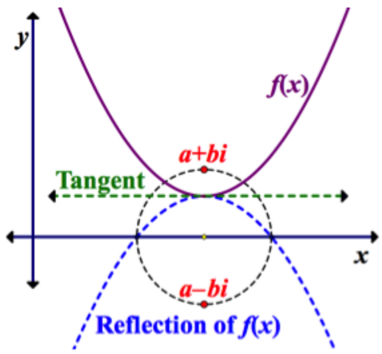


Figure 1: Complex Roots of a Quadratic

Earlier we defined analytic constructions as any combination of geometric constructions and analytic calculations based only on observable values. More precisely, we can define these values as geometric features that secondary students can easily recognize and estimate such as the location of real roots, points of inflection, points of tangency or bitangency, and values of the function.

In the following sections we consider quadratic, cubic, quartic, and special polynomials of higher degree in successive order. We hope readers will take the time to experiment with the interactive dynamic figures.

## 2 Quadratic Polynomials

Begin with the real, monic, quadratic polynomial, $f(x)$, with complex roots. Construct vertical line $L$ through the vertex of $f(x)$ and denote the $x$-intercept of $L$ as $a$. Construct circle $C$ centered at the origin with radius $R = \sqrt{f(0)}$. The intersections of $L$ and $C$ are the complex roots $(a, \pm b\imath)$.

Experiment with Dynamic Figure 2 to experience this concept. (*Click the image to access the Dynamic Figure.*) Notice that, even when you ask the circle and line to show, these elements do not exist when the complex roots become real and the conjugate pair becomes a double real root. The Axes slider (upper slider) dilates the coordinate plane to zoom in and out. The Scale slider (lower slider) multiplies the value of the function by a factor from 0 to 1.

## 3 Cubic Polynomials

Begin with the real, monic, cubic polynomial, $g(x)$, with a real root at $r$ and two non-real roots. Construct tangent $T$ from the real root to the curve $g(x)$. Denote the point of tangency $P = (a, g(a))$. (Note that if $g''(a) = 0$, then $a = -r/2$.) Construct a vertical line through $P$ which intercepts the $x$-axis at $(a, 0)$. Construct a circle $C$ centered at the origin with radius $R = \sqrt{g(0)/(-r)}$. (Note that if $r = 0$, then $R = \sqrt{g'(0)}$.) The intersections of $L$ and $C$ are the complex roots $(a, \pm b\imath)$.

Experiment with Dynamic Figure 3 (*click the image*) to experience this concept. Notice that, even when you ask the circle and line to show, these elements do not exist when the complex roots become real and the conjugate pair becomes a double real root.

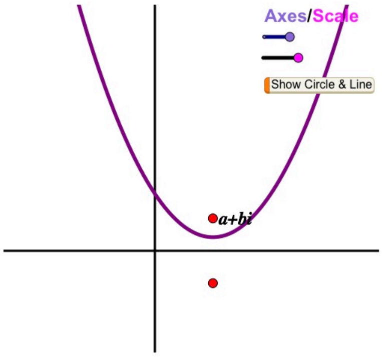


Figure 2: Dynamic Figure – Quadratic Polynomials

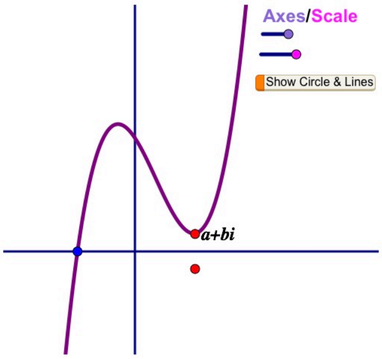


Figure 3: Dynamic Figure – Cubic Polynomials

# 4 Quartic Polynomials

In order to investigate quartic polynomials with complex roots, we must consider two cases: quartics with two real roots and quartics with no real roots. Notice that in the following discussions significantly more analytic equations are provided than for the quadratic and cubic cases. However, in keeping with the theme of this investigation, all values in these expressions are directly observable from the respective graph.

## 4.1 Two Real Roots

Begin with a real, monic, quartic polynomial, $h(x)$, with two real roots, $r_1$ and $r_2$, where $r_1$ and $r_2$ may be equal. A student will collect the following values from the graph: $r_1$ and $r_2$, $h(0)$ and $h'(0)$, and, if needed, $h''(0)$. Employ the appropriate condition to determine values for $a$ and $R$.

If $r_1 \neq 0 \neq r_2$, then

$$a = \frac{-1}{2r_1 r_2}\left[h(0)\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+h'(0)\right] \quad \text{and} \quad R = \sqrt{\frac{h(0)}{r_1 r_2}}.$$

If $r_1 = 0 \neq r_2$, then

$$a = \frac{1}{2r_2}\left[\frac{h'(0)}{r_2} + \frac{h''(0)}{2}\right] \quad \text{and} \quad R = \sqrt{\frac{-h'(0)}{r_2}}.$$

If $r_1 = 0 = r_2$, then (with $h''(0)$ also read from the graph)

$$a = h(1) - \frac{1}{2}h'(1) + 1 \quad \text{and} \quad R = \sqrt{\frac{h''(0)}{2}}.$$

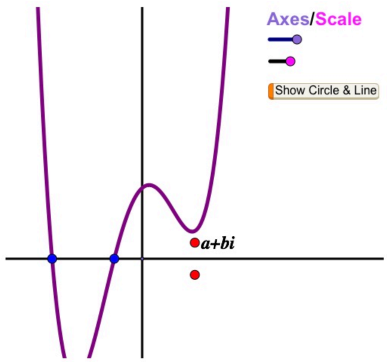


Figure 4: Dynamic Figure – Quartic Polynomials

In the case of two distinct, nonzero real roots, we used only four observed values to determine the complex roots; in the other cases, we were able to use fewer points. If we had used Lagrange interpolation (also based on observable values) to determine the equation of the quartic, we would have needed to use five values from the graph. Additionally, Lagrange interpolation would not have been consistent with the theme of constructing circles and lines to visually identify complex roots.

## 4.2 Two Real Roots: Alternate Technique Employing a Bitangent or Inflection Points

Begin with a real, monic, quartic polynomial, $h(x)$, with two real roots, $r_1$ and $r_2$, where $r_1$ and $r_2$ may be equal. However, include the additional condition that there exists a bitangent to $h(x)$. (See Figure 5 for an example of a quartic with a bitangent, a line tangent to a function at two points.) There exists a bitangent on $h(x)$ through points $(x_{b_1}, h(x_{b_1}))$ and $(x_{b_2}, h(x_{b_2}))$ with inflection points $(x_{i_1}, h(x_{i_1}))$ and $(x_{i_2}, h(x_{i_2}))$ iff for the complex roots $a \pm b\imath$ of $h(x)$, $b^2 < \frac{1}{2}\left(a - \frac{1}{2}(r_1 + r_2)\right)^2 + \frac{1}{4}(r_1+r_2)^2 - r_1 r_2$. Note that if a bitangent exists, it is unique. Calculate

$$a = \left[(x_{b_1} + x_{b_2}) - \tfrac{1}{2}(r_1 + r_2)\right] = \left[(x_{i_1} + x_{i_2}) - \tfrac{1}{2}(r_1 + r_2)\right]$$

and

$$R = \sqrt{-\frac{h'(0) + 2ar_1r_2}{r_1 + r_2}}.$$

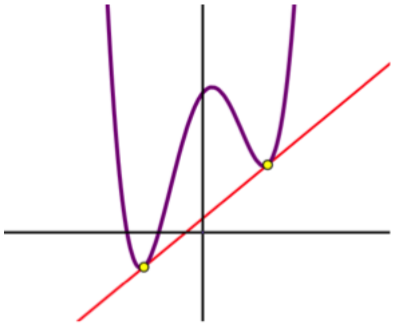

Figure 5: Quartic with Bitangent

Construct the vertical line $L$ through $(a, 0)$ and circle $C$ centered at the origin with radius $R$. The intersections of $C$ and $L$ are the complex roots $a \pm b\imath$.

Note that in previous cases, the observable values selected were primarily the values of real roots. In this case, we have also included points of bitangency and inflection, both of which remain recognizable by secondary students.

Experiment with Dynamic Figure 6 to experience this concept. (*Click the image.*) Notice that in addition to the circle and line constructions, the existence of the bitangent is contingent upon conditions provided above.

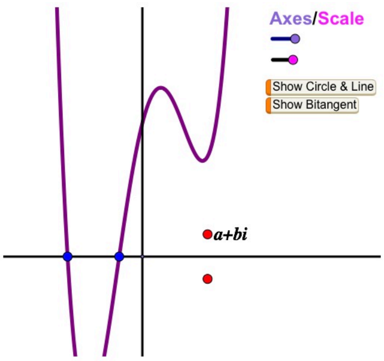


Figure 6: Dynamic Figure – Quartic Polynomials and Bitangents

### 4.3 No Real Roots

Begin with a real, monic, quartic symmetric polynomial, $h(x) = ((x-a)^2 + b_1^2)((x+a)^2 + b_2^2)$, with no real roots and complex roots $a \pm b_1\imath$ and $a \pm b_2\imath$. (A technique is provided below to translate a non-symmetric quartic with roots $a \pm b\imath$ and $c \pm d\imath$ to symmetric form.) This quartic, can be rewritten as the depressed quartic, $h(x) = x^4 + \frac{1}{2}h''(0)x^2 + h'(0)x + h(0)$.

In the geometric view, a formula for $h$ is not available; however, using the observable values $y_0 = h(0)$, $y_0' = h'(0) = \frac{1}{2}(h(1) - h(-1))$, and $y_0'' = h''(0) = h(1) - 2h(0) + h(-1) - 2$, we can calculate the sequence

$$
\begin{aligned}
p_1 &= \tfrac{1}{4}y_0''^3 + 27{y_0'}^2 - 36\, y_0''\, y_0,\\
p_2 &= p_1 + 2\sqrt{\tfrac{1}{4}\,p_1^2 - \left(\tfrac{1}{4}\,y_0''^2 + 12\,y_0\right)^3},\\
p_3 &= \frac{\tfrac{1}{4}\,y_0''^2 + 12\,y_0}{3\sqrt[3]{\tfrac{1}{2}p_2}} + \tfrac{1}{3}\sqrt[3]{\tfrac{1}{2}p_2},\\
p_4 &= \sqrt{p_3 - \tfrac{1}{3}\,y_0''}\,,\\
p_5 &= -\tfrac{2}{3}\,y_0'' - p_3,\\
p_6 &= -2\,\frac{y_0'}{p_4}.
\end{aligned}
$$

The $p$ values computed determine the circle and line showing the complex roots of $h$ as follows. For $a = \frac{1}{2}p_4$, construct a vertical line $L_1$ through $(a, 0)$ and a line $L_2$ through $(-a, 0)$. Centered at the origin, construct circles $C_1$ with radius $R_1 = \frac{1}{2}\sqrt{p_4^2 + p_5 + p_6}$ and $C_2$ with radius $R_2 = \frac{1}{2}\sqrt{p_4^2 + p_5 - p_6}$. (Proof of this construction is provided in [3].) The complex roots are located at the intersections of $L_1$ and $C_1$ and the intersections of $L_2$ and $C_2$. Experiment with Dynamic Figure 7 to experience this concept. (*Click the image.*) Notice that this produces two sets of lines and circles to locate the complex roots.

In the dynamic figure below, we investigate a general monic fourth degree polynomial with non-symmetric roots $a + b\imath$ and $c + d\imath$.

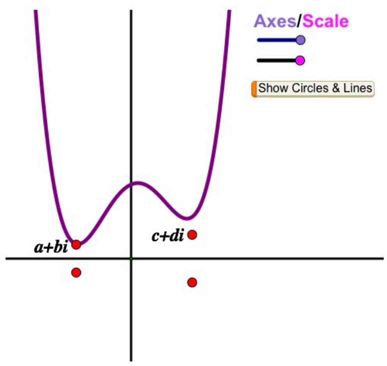


Figure 7: Dynamic Figure – Quartic Polynomials with Two Complex Root Pairs

## 4.4 Transforming Quartics to Symmetric Form

Situations arise when a quartic needs to be translated to symmetric form. In the case of quartics with two real roots, the symmetric form places the graph of the quartic such that the $y$-axis is at the mean of the two real roots. For quartics with no real roots, the symmetric form places the graph of the quartic such that the $y$-axis is at the mean of the real parts of the two complex roots.

### 4.4.1 Two Real Roots

To convert a monic quartic polynomial having two real roots to symmetric form the following transformation is applied. For $h(x) = (x-r_1)(x-r_2)\left((x-a)^2+b^2\right)$, use $t = x-\frac{1}{2}(r_1+r_2)$, $r = \frac{1}{2}(r_1-r_2)$, and $A = a - \frac{1}{2}(r_1+r_2)$. Then the translated $h(x)$ is $h(t) = (t-r)(t+r)\left((t-A)^2+b^2\right)$ now in symmetric form.

### 4.4.2 No Real Roots

To convert a monic quartic polynomial having no real roots to symmetric form the following transformation is applied. For $h(x) = \left((x-a)^2+b^2\right)\left((x-c)^2+d^2\right)$, use $t = x-\frac{1}{2}(a+c)$ and $e = \frac{1}{2}(a-c)$, then the translated $h(x)$ is $h(t) = \left((t-e)^2+b^2\right)\left((t+e)^2+d^2\right)$ which is now in symmetric form.

## 5 Special Higher Degree Polynomials

Let us consider a real, monic polynomial with one real root $r$ of multiplicity $n \in \mathbb{N}$ and a simple complex conjugate root pair, $a \pm b_1\imath$. This can be written as

$$j(x) = (x-r)^n\left((x-a)^2+b^2\right).$$

Graph the auxiliary function

$$\hat{\jmath}(x) = j'(x) - nj(x)/(x-r).$$

The auxiliary function $\hat{\jmath}$ is a generalization of the tangent line developed in the classical construction of the complex roots of a cubic polynomial (see, for example, [10] or [6]).

Locate the non-$r$ real root of $\hat{\jmath}$ and denote it as $x = a$. Construct vertical line $L$ through $(a, 0)$. Calculate the radius $R = \sqrt{|\,j(0)/r^n|}$. (If $r = 0$, then calculate $R = \sqrt{|\,j^{(n)}(0)/n!|}$.) Construct a circle $C$ centered at the origin with radius $R$. The intersections of $L$ and $C$ are the complex roots $a \pm b\imath$.

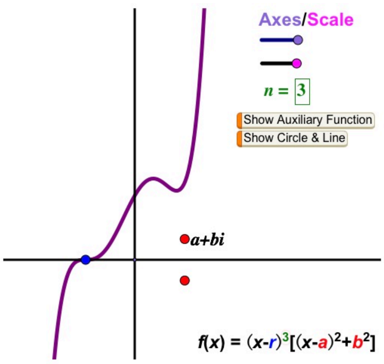


Figure 8: Dynamic Figure – Special Higher Degree Polynomials

# 6 Summary

To develop this investigation, a flowchart depicting the decision tree regarding locating the complex roots of symmetric, real, monic, quartics with no real roots was developed. (This decision tree can be viewed by clicking: *Quartic Flowchart*.) Notably, this decision tree is nontrivial. However, when the decision tree is paired with the dynamic graphing technology, the mathematics comes alive. Furthermore, using Boolean conditions in the applets allows for the avoidance of many cases in which processes are undefined. This helps students experience the associated mathematics without glitches and technological hiccups.

Therefore, answering the initial question, yes, the complex roots of quadratics, cubics, and quartics can all be located using analytic constructions with lines and circles. It is hoped that providing this information in the form of a dynamic HTML document and allowing students to interact with the applets facilitates understanding, learning, and engagement.

# Acknowledgements

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